Journal of Pure and Applied Algebra 28 (1983) 249-264
North-Holland

STRONG S-DOMAINS

S. MALIK and J.L. MOTT
Department of Mathematics, Florida State University, Talahassee, FL 32306, USA

Communicated by H. Bass
Received 8 November 1981

S-domains and strong S-rings are studied extensively with special emphasis on integral and polynomial ring extensions. The main theorem of this paper is that for a Prüfer domain R, the polynomial ring $R[X_1, \dots, X_n]$ in finitely many indeterminates is a strong S-domain. We also prove that any Prüfer v-multiplication domain is an S-domain.

1. Introduction and terminology

All rings under consideration are commutative rings with unity. The concepts of S-domain and strong S-domain are crucial ones and were introduced by Kaplansky [11, p. 26]. Let us recall their definitions. An integral domain R is an S-domain if for each prime ideal P of R of height one the extension $PR[X]$ to the polynomial ring in one variable is also of height one. Call a ring R a strong S-ring if the residue class ring R/P is an S-domain for each prime P of R.

The present paper deals with several elementary properties of strong S-domains and the behaviour of the strong S-property under integral and polynomial ring extensions.

In Section 2 we first prove that the strong S-property is a local property. Then using this result and Theorem 68 of [11] we see immediately that a Prüfer domain is a strong S-domain. One reason that Kaplansky introduced the notion of strong S-domain was to treat the classes of Noetherian domains and Prüfer domains in a unified manner - for if R is either a Noetherian or a Prüfer domain then R is a strong S-domain. Moreover, if R is in either of the two classes of domains, then the following dimension formula holds: $\dim R[X_1, \dots, X_n] = n + \dim R$. Kaplansky observed that for $n = 1$ and for R a strong S-domain then $\dim R[X_1] = 1 + \dim R$. Had the strong S-property been stable under polynomial ring extensions, the above dimension formula could have been obtained by induction for all strong S-domains. However, the strong S-property is not stable, in general, and thus by itself is not the cause for the dimension formula. Nevertheless, we show in Theorem 3.5 that if R is a Prüfer domain, then $R[X_1, X_2, \dots, X_n]$ is a strong S-domain. Hilbert's Basis Theorem and connected results give the corresponding result for Noetherian

0022-4049/83/$03.00

domains. Thus we define a ring R to satisfy the stably strong S-property, if for each n, $R[X_1, X_2, \ldots, X_n]$ is a strong S-ring. Thus, Noetherian rings and Prüfer domains are better unified under the concept of stably strong S-property, and the stably strong S-property does in fact imply the above dimension formula.

Using a theorem of Nagata [14] we also prove that if R is a Prüfer domain, then for any prime ideal P of $R[X_1, X_2, \ldots, X_n]$ of finite height, ht P = little rank P and $R[X_1, X_2, \ldots, X_n]$ satisfies the saturated chain condition [11, p. 99].

In view of above conclusions we then ask what properties common to Noetherian domains and Prüfer domains cause the stability of strong S-property under polynomial ring extensions. One property that these domains have in common is that their integral closures are Prüfer v-multiplication domains (Prüfer domains are already integrally closed and the integral closure of a Noetherian domain is a Krull domain). Then the natural question arises: Could the stably strong S-property be caused by the property that the integral closure of a domain R is a Prüfer v-multiplication domain? The answer is no in general for we give an example of Krull domain that is not a strong S-domain. Nevertheless we show in Theorem 4.16 that such a domain at least must be an S-domain. We then use this theorem to conclude in Proposition 4.19 and 4.21 that if R is either Noetherian or Prüfer then the integral closure of R is a stably strong S-domain. Thus the two classes are further unified by this observation.

The ultimate effect of this last observation is to focus the study of stably strong S-property onto the class of Krull domains, a subject we leave for future research.

In Section 5 we study the '$D+M$' construction [9] and other related constructions that inherit the strong S-property from D.

2. Elementary properties

We now give some elementary properties of S-domain and strong S-rings. Note that it is immediate that the direct sum of any finite number of rings is a strong S-ring if and only if each summand is.

Proposition 2.1. *A domain R is an S-domain if and only if R_M is an S-domain for each maximal ideal M of R.*

Proof. Let R be an S-domain. For any maximal ideal M of R, let $P^e = PR_M$ be a height 1 prime of R_M. Clearly P is a height 1 prime ideal of R. But then R is an S-domain, hence $PR[X] = P^*$ is a height 1 prime of $R[X]$. Now

$$P(R[X]]_{R\setminus M} = PR_M[X] = P^*R_M[X] = (PR_M)R_M[X] = P^eR_M[X],$$

so $P^eR_M[X]$ is a height 1 prime of $R_M[X]$ and R_M is an S-domain.

Conversely, let R_M be an S-domain for each maximal ideal M of R and let P be a height 1 prime of R contained in a maximal ideal M. Then PR_M is a height 1 prime of R_M. But then R_M is an S-domain, hence $P^e R_M[X] = (PR_M)R_M[X] = PR_M[X]$ is a height 1 prime of $R_M[X]$ and by taking intersections with $R[X]$ we have $PR[X]$ is a height 1 prime of $R[X]$ and hence by definition R is an S-domain.

Corollary 2.2. *The following are equivalent in a domain R:*

(i) *R is an S-domain.*

(ii) *R_S is an S-domain for each multiplicative system S of R.*

(iii) *R_P is an S-domain for each prime ideal P of R.*

(iv) *R_M is an S-domain for each maximal ideal M of R.*

Proof. We prove (i) implies (ii), (ii) implies (iii), (iii) implies (iv) and (iv) implies (i). So let $P_S = PR_S$ be a prime ideal of R_S of height 1 then P is a prime ideal of R of height 1 such that $P \cap S = \emptyset$. Since R is an S-domain it follows $PR[X]$ is a height 1 prime ideal of $R[X]$. Therefore $P_S R_S[X] = (PR[X])_S$ is also a prime ideal of height 1 in $(R[X])_S = R_S[X]$. So R_S is an S-domain proving thereby (i) implies (ii).

To prove (ii) implies (iii) take $S = R \setminus P$ where P is any prime ideal of R.

Clearly (iii) implies (iv) since each maximal ideal is a prime ideal and (iv) implies (i) follows from Proposition 2.1.

Proposition 2.3. *A ring R is a strong S-ring if and only if R_M is a strong S-ring for each maximal ideal M of R.*

Proof. Since $R_M/PR_M = (R/P)_{\bar{M}}$ where $\bar{M} = (R \setminus M) + P/P$, P being a prime ideal of R contained in M, it is enough to prove that R is an S-domain if and only if R_M is an S-domain for each maximal ideal M of R but then Proposition 2.1 completes the proof.

Corollary 2.4. *The following are equivalent for any ring R:*

(i) *R is a strong S-ring*

(ii) *R_S is a strong S-ring for each multiplicative system S of R*

(iii) *R_P is a strong S-ring for each prime P of R*

(iv) *R_M is a strong S-ring for each maximal ideal M of R.*

In view of above corollary and Theorem 68 [11] the following proposition is immediate.

Proposition 2.5. *A Prüfer domain is a strong S-ring.*

Proposition 2.6. *Let R be a domain, then R is a strong S-ring if and only if each flat overring of R is strong S-ring.*

Proof. Let T be a flat overring of R and suppose that R is a strong S-ring. For each maximal ideal M of T, $T_M = R_{M \cap R}$ [16]. Since R is a strong S-ring, $R_{M \cap R}$ is also a strong S-ring so T_M is a strong S-ring. But then by Proposition 2.3 T is a strong S-ring.

The converse follows by Corollary 2.4.

Proposition 2.7. *Let R be a domain. Suppose R is a strong S-ring, and $R_1, \ldots, R_n$ are quasi-semi-local flat overrings of R contained in the quotient field of R; then $R' = \bigcap_{i=1}^{n} R_i$ is a strong S-ring.*

Proof. Since each R_i is a flat overring of R, by Proposition 2.6 R_i is a strong S-ring. Also $R \subseteq R' \subseteq R_i \subseteq K$ and since R_i are flat overrings of R, it follows that R_i are flat overrings of R'. Now every nonunit of R' is a nonunit in some R_i. Thus the set of nonunits of R' is exactly the union of the finite set of contracted maximal ideals of R_i for $1 \leq i \leq n$. If M is any maximal ideal of R and $x \in M$ then $x \in M_i$ for some maximal ideal M_i of some R_i. Hence $x \in M_i \cap R'$ so $M \subseteq \bigcup (M_{i_k} \cap R')$; but R_i are semi-local, the union $\bigcup (M_{i_k} \cap R')$ is a finite union so that $M \subseteq M_{i_k} \cap R'$ for some i_k. But then M was maximal so $M = M_{i_k} \cap R'$, showing that each maximal ideal of R' is a contraction of some maximal ideal of R_i. Let M be any maximal ideal of R. If $M = M_i \cap R'$, M_i being a maximal ideal of R_i, then $R_{i_{M_i}} = R'_{M_i \cap R} = R'_M$ as R_i is a flat overring of R'. Therefore, as each R_i is a strong S-ring, $R_{i_{M_i}}$ is a strong S-ring and R'_M is a strong S-ring. The conclusion now is immediate from Proposition 2.3.

The proofs of Proposition 2.8 and Corollary 2.9 are easy applications of Proposition 1.1 of [2] and Propositions 2.3 and 2.7.

Proposition 2.8. *If R is a non-quasi-local domain, then R is a strong S-ring if and only if $T(x)$, the integral transform of (x) for each nonunit x of R, is a strong S-ring.*

Corollary 2.9. *Let R be a domain and A be a finitely generated ideal of R. Then if R is a strong S-ring the integral transform $T(A)$ is also a strong S-ring.*

3. Strong S-rings and polynomial extensions

In this section we prove necessary and sufficient conditions for the polynomial ring $R[X_1, \ldots, X_n]$ to inherit the S-property or the strong S-property. In particular, we prove that if R is a Prüfer domain, then $R[X_1, \ldots, X_n]$ is a strong S-domain.

Theorem 3.1. *Let R be an S-domain and $X = \{X_1, X_2, \ldots, X_n\}$ be a finite set of indeterminates over R. Then $R[X]$ is an S-domain if and only $R_P[X]$ is an S-domain for each prime ideal P of R.*

Proof. Let $R[X]$ be an S-domain, then $R_P[X] = (R[X])_{R \setminus P}$ for each prime ideal P

of R is a quotient ring of $R[X]$, hence, is an S-domain.

Conversely, let $R_P[X]$ be an S-domain for each prime ideal P of R and Q be a height 1 prime of $R[X]$. If $Q \cap R = P$ two cases arise according as P is nonzero or zero. Suppose first $P \neq (0)$, then $R[X]_{P[X]} \supseteq R[X]_{R \setminus P} = R_P[X]$. As $P[X] \subseteq Q$ and height of Q is 1, so $P[X] = Q$. Therefore $R[X]_Q = R[X]_{P[X]}$ is a quotient ring of $R_P[X]$ and hence an S-domain. Thus $QR[X]_Q[Y] = QR[X, Y]_Q$ is a prime ideal of $R[X, Y]_Q$ of height 1. Consequently, $Q[Y] = QR[X, Y]_Q \cap R[X, Y]$ is a prime ideal of $R[X, Y]$ of height 1 and $R[X, Y]$ is an S-domain. If $P = (0)$ then $R_P[X] = K[X]$, where K is a quotient field of R, is Noetherian and hence an S-domain. Also $QK[X]$ is a prime ideal of $K[X]$ of height 1. Thus $QK[X, Y]$ is also of height 1 in $K[X, Y]$. It now follows that $Q[Y] = Q[X, Y] \cap R[X, Y]$ is a height 1 prime of $R[X, Y]$.

Theorem 3.2. *Let R be a strong S-domain and $X = \{X_1, X_2, \ldots, X_n\}$ a finite set of indeterminates over R. Then $R[X]$ is a strong S-ring iff $R_P[X]$ is a strong S-ring for each prime ideal P of R.*

Proof. If $R[X]$ is a strong S-ring, then for each prime ideal P of R, $R_P[X] = R[X]_{R \setminus P}$ is a quotient ring of $R[X]$ and hence a strong S-ring.

Conversely, suppose that $R_P[X]$ is a strong S-ring for each prime ideal P of R and $Q_1 \subset Q_2$ be a pair of adjacent primes of $R[X]$. Also let $Q_i \cap R = P_i$ for $i = 1$ and 2; then $Q_2 \cap (R \setminus P_2) = \emptyset$ and $Q_1 R_{P_2}[X] \subset Q_2 R_{P_2}[X]$ is a pair of adjacent primes of $R_{P_2}[X]$. But $R_{P_2}[X]$ is a strong S-ring so that $Q_1 R_{P_2}[X][Y] \subset Q_2 R_{P_2}[X][Y]$ is a pair of adjacent primes of $R_{P_2}[X][Y]$. Once again we obtain $Q_1[Y] \subset Q_2[Y]$ are adjacent primes of $R[X, Y]$. Hence $R[X]$ is a strong S-ring.

Theorem 3.3. *Suppose that R is an integral domain and $X = \{X_1, X_2, \ldots, X_n\}$ a finite set of indeterminates over R. Then $R[X]$ is an S-domain if and only if $R[X]_{M[X]}$ is an S-domain for each maximal ideal M of R (that is, if and only if $R(X)$ is an S-domain).*

Proof. Since $R[X]_{M[X]}$ is a quotient ring of $R[X]$, if $R[X]$ is an S-domain, $R[X]_{M[X]}$ is an S-domain. Conversely, let Q be a prime ideal of height 1 in $R[X]$. If $Q \cap R = (0)$ then $QK[X]$ is a height 1 prime of $K[X]$, where K is the quotient field of $R[X]$. But $K[X]$ is a Noetherian domain, hence an S-domain. Thus $QK[X][Y]$ is a height 1 prime ideal of $K[X][Y]$. But then $QK[X][Y] \cap R[X, Y] = Q[Y]$ is a height 1 prime of $R[X, Y]$. If $Q \cap R = P$, then $P \subseteq M$ for some maximal ideal M of R. Now since Q is a height 1 prime $Q = P[X]$ and $PR[X] \subseteq M[X]$, it follows that $QR[X]_{M[X]}$ is a height 1 prime ideal of $R[X]_{M[X]}$ which is an S-domain, so $QR[X]_{M[X]}[Y]$ is a height 1 prime of $R[X]_{M[X]}[Y]$. But then $QR[X]_{M[X]}[Y] \cap R[X, Y] = Q[Y] = Q[Y]$ is also a height 1 prime of $R[X, Y]$. So $R[X]$ is an S-domain.

The last conclusion follows immediately from the fact that the maximal ideal of $R(X)$ are of the form $M(X)$ when M is a maximal ideal of P and also that $R[X]_{M[X]} = R(X)_{M(X)}$.

Corollary 3.4. *Let R be a Prüfer domain, then* $R[X]$ *is an S-domain.*

Proof. Since R is a Prüfer domain, $R(X)$ is a Prüfer domain by Proposition 33.4 of [7]. Now for each maximal ideal M of R $(R(X)_{M(X)}=R[X]_{M[X]}$ so $R[X]_{M[X]}$ is a strong S-domain and hence an S-domain, but then by Theorem 3.3 the conclusion follows.

One sees readily that $R(X)$ is a strong S-domain if and only if $R[X]_{M[X]}$ is a strong S-domain for each maximal ideal M of R. But then the following question comes to mind: Is $R[X]$ a strong S-domain if and only if $R(X)$ is a strong S-domain?

It is very clear that if $R[X]$ is a strong S-ring, $R(X)$ is a strong S-ring but it is in other direction that deep waters run. For any pair $Q_1 \subset Q_2$ of adjacent primes of $R[X]$ we are unable to prove that $Q_1[Y] \subset Q_1[Y]$ are adjacent primes of $R[X, Y]$ without any condition on R. In fact we show that the condition that R is a Prüfer domain is sufficient.

Theorem 3.5. *Let R be a Prüfer domain and let* $X=\{X_1, \ldots, X_n\}$ *be a finite set of indeterminates over R. Then* $R[X]$ *is a strong S-domain.*

Proof. Let $Q_1 \subset Q_2$ be a adjacent primes in $R[X]$ and let $P_i=Q_i \cap R$. Let $\bar{R}=R/P_1$, $\bar{P}_2=P_2/P_1$, $S=\bar{R}\setminus\bar{P}_2$, $T=R\setminus P_2$ and $V=R_T/P_1R_T$.

Then $(\bar{R}[X])_S=(\bar{R})_S[X]=V[X]$.

Now V is a valuation ring since R is a Prüfer domain. But, more than that, we claim that $\dim V \le 1$. This follows since $Q_2/Q_1 \cap R=P_2$ and the pair $(R/P_1, R[X]/Q_1)$ has the going down property because R/P_1 is a Prüfer domain. Therefore, $\operatorname{ht}(P_2/P_1) \le 1$ since $\operatorname{ht}(Q_2/Q_1)=1$.

Next, if we know that $V[X]$ is a strong S-domain, the proof of the theorem would be complete. For the prime ideals $\bar{Q}_i=Q_i/P_1[X]$ in $R[X]/P_1[X]=\bar{R}[X]$ lift to adjacent prime ideals in $V[X]$ since $Q_2 \cap T=\emptyset$. But $V[X]$ a strong S-domain implies that $\bar{Q}_1V[X, Y] \subset \bar{Q}_2V[X, Y]$ are adjacent primes in $V[X, Y]$. But then it is immediate that $Q_1R[X, Y] \subset Q_2R[X, Y]$ must be adjacent primes of $R[X, Y]$.

Therefore, let us show that if V is a valuation ring of rank 1 (that is, $\dim V=1$) then $V[X]$ is a strong S-domain (note the 0-dimensional case is obvious).

Let $Q_1 \subset Q_2$ be adjacent primes of $V[X]$ and $P_i=Q_1 \cap V$. We may assume $Q_1 \ne (0)$ for otherwise $\operatorname{ht} Q_2=1$ and, since $V[X]$ is an S-domain by Corollary 2.13, $\operatorname{ht} Q_2[Y]=1$ in $V[X, Y]$. Moreover, we may assume $P_2 \ne (0)$ since otherwise Q_1 and Q_2 would lift to adjacent primes in the strong S-domain $K[X]$ and, in turn, $Q_1K[X, Y]$ and $Q_2K[X, Y]$ are adjacent primes in $K[X, Y]$. But then $Q_1V[X, Y]$ and $Q_2V[X, Y]$ must be adjacent primes in $V[X, Y]$.

Thus, we have reduced to proving the theorem in the case that V is valuation ring of dimension one, $Q_1 \subset Q_2$ are adjacent primes of $V[X]$ such that $Q_1 \cap V=(0)$ and $Q_2 \cap V=M$, the maximal ideal of V.

A consequence of Nagata's theorem [14] is that all saturated chains of prime ideals of $V[X,Y]$ between $Q_2[Y]$ and (0) have the same length. Thus, $Q_1[Y]$ and $Q_2[Y]$ will be adjacent primes if and only if $\operatorname{ht} Q_2[Y] = 1 + \operatorname{ht} Q_1[Y]$. Let us use Nagata's theorem to determine each height:

$$\begin{aligned}\operatorname{ht} Q_2[Y] &= \operatorname{ht} M + \operatorname{tr} \deg_V V[X,Y] - \operatorname{tr} \deg_{V/M} V[X,Y]/Q_2[Y] \\ &= 1 + (n+1) - (\operatorname{tr} \deg_{V/M} V[X]/Q_2 + 1) \\ &= n + 1 - \operatorname{tr} \deg_{V/M} V[X]/Q_2, \\ \operatorname{ht} Q_1[Y] &= \operatorname{tr} \deg_V V[X,Y] - \operatorname{tr} \deg_V V[X,Y]/Q_1[Y] \\ &= n - \operatorname{tr} \deg_V V[X]/Q_1.\end{aligned}$$

Thus, we see that $Q_1[Y]$ and $Q_2[Y]$ are adjacent primes if and only if

$$\operatorname{tr} \deg_V V[X]/Q_1 = \operatorname{tr} \deg_{V/M} V[X]/Q_2.$$

To prove this equality we apply Nagata's theorem again. Let $A = V[X]/Q_1$ and $P = Q_2/Q_1$. Since $Q_1 \cap V = (0)$, V can be embedded in A and $P \cap V = M$. Since Q_1 and Q_2 are adjacent primes of $V[X]$,

$$\begin{aligned}\operatorname{ht} P &= 1 \\ &= \operatorname{ht} M + \operatorname{tr} \deg_V A - \operatorname{tr} \deg_{V/M} A/P \\ &= 1 + \operatorname{tr} \deg_V V[X]/Q_1 - \operatorname{tr} \deg_{V/M} V[X]/Q_2.\end{aligned}$$

In other words, the two transcendence degrees are equal, $Q_1[Y]$ and $Q_2[Y]$ are adjacent primes in $V[X,Y]$, and the proof is complete.

Corollary 3.6. *Suppose that R is a Prüfer domain. Then any finitely generated extension* $R[a_1, \ldots, a_n]$ *is a strong S-ring.*

Proof. The ring $R[a_1, \ldots, a_n]$ is a homomorphic image of $R[X_1, \ldots, X_n]$.

Definition 3.7. The little rank of a prime ideal P of a ring R is the length of the shortest saturated chain descending from P to a minimal prime of R.

Let us extend Nagata's theorem [14] slightly.

Theorem 3.8. *Let R be a Prüfer domain and Q be a prime ideal of* $R[X_1, \ldots, X_n]$ *of finite height, then* little rank $Q = \operatorname{ht} Q$.

Proof. Let $Q \cap R = P$, then $\operatorname{ht} P$ is also finite, since

$$\begin{aligned}\operatorname{ht} Q &= \operatorname{ht} P[X_1, \ldots, X_n] + \operatorname{ht}(Q/P[X_1, \ldots, X_n]) \quad \text{by Theorem 1 [3]} \\ &\geq \operatorname{ht} P[X_1, \ldots, X_n] = \operatorname{ht} P.\end{aligned}$$

Now to prove that little rank $Q = \mathrm{ht}\, Q$ it is enough to prove

$$\text{little rank } QR_P[X_1, \ldots, X_n] = \mathrm{ht}\, QR_P[X_1, \ldots, X_n].$$

Hence we may assume R is a valuation ring of finite rank. But now by Nagata's theorem we get the desired conclusion.

Corollary 3.9. *Under the hypothesis of Theorem* 3.8 *all saturated chains of prime ideals of* $R[X_1, \ldots, X_n]$ *descending from* Q *to* (0) *have the same length.*

Consequently, if R is Prüfer domain in which each prime ideals has finite height, then $R[X_1, \ldots, X_n]$ satisfies the saturated chain condition, that is, if $P \subset Q$ are prime ideals of $R[X_1, \ldots, X_n]$, then all saturated chains of prime ideals between P and Q have the same length.

Definition 3.10. If R is a domain and k is a nonnegative integer such that for each valuation overring V of R dim $V \leq k$ and there exists at least one valuation overring whose dimension is exactly equal to k, then we say that R has valuative dimension k and write $\dim_V R = k$. If no such k exists, then we say that $\dim_V R = \infty$.

It is well known that $\dim R \leq \dim_V R$. Moreover, if $R[X_1, \ldots, X_n]$ is a strong S-ring for each positive integer n then $\dim R = \dim_V R$ provided $\dim_V R < \infty$. The converse in general is false for there exist domains R for which $\dim R = \dim_V R$ but R is not a strong S-domain. We give the following example.

Example 3.11. If R has finite dimension n_0 and for each positive integer m, n_m denotes the dimensions of $R[X_1, \ldots, X_n]$, then the sequence $(n_i)_{i=0}^{\infty}$ is called the dimension sequence of R and the sequence $\{d_i\}_{i=1}^{\infty}$, where $d_i = n_i - n_{i-1}$ is called the difference sequence for R. Denote by $\mathcal{S}$ the set of sequences $s = \{n_i\}_{i=0}^{\infty}$ of nonnegative integers such that the associated difference sequence $\{d_i\}_{i=1}^{\infty}$ satisfies $1 \leq d_{i+1} \leq d_i \leq n_0 + 1$. For $s_1, \ldots, s_r \in \mathcal{S}$. $s_i = \{n_j^{(i)}\}_{j=0}^{\infty}$, $\sup\{s_1, \ldots, s_r\}$ is defined to be the sequence $s = \{n_j\}_{j=0}^{\infty}$, where $n_j = \sup\{n_j^{(1)}, \ldots, n_j^{(r)}\}$ for each $j \geq 0$.

In [1], it is proved that given a dimension sequence s and field K there is a domain R with quotient field K and dimension sequence s. A method to construct a semi-quasi-local domain that has s as its dimension sequence is also given. Following this we construct the sequences $\{1, 3, 4, 5, \ldots\}$ and $\{3, 4, 5, 6, \ldots\}$ in $\mathcal{S}$. By Lemma 4.7 and Proposition 4.8 in [1], there exist domains $J_1 = R_1 + M_1$ and $J_2 = R_2 + M_2$ with respective dimension sequence $\{1, 3, 4, 5, \ldots\}$ and $\{3, 4, 5, 6, \ldots\}$. Set $R = J_1 \cap J_2$ then by Theorem 4.10 in [1], J_1 and J_2 are quotient rings of R and $s = \{3, 4, 5, 6, \ldots\}$ is the dimension sequence of R. Here $\dim R = 3$ and $\dim R[X_1, X_2, X_3] = 6$ so that $\dim_V R = 3$ [7]. As J_1 is a quotient ring of R and not a strong S-ring, R cannot be a strong S-ring. The reason why J_1 fails to be a strong S-ring is because $\dim J_1[X_1] = 3$ and if J_1 were to be a strong S-ring $\dim J_1[X_1]$ would have to be 2.

4. Strong S-rings and integral extensions

Here we study various conditions on the domain R or on its integral extension T to study ascent and descent of the strong S-property. The proof of our next lemma is an easy application of the incomparability property of integral extensions and is therefore omitted.

Lemma 4.1. *Let T be an integral extension of a domain R and $P_1 \subset P_2$ be a pair of adjacent primes of R. If Q_i is a prime ideal of T such that $Q_i \cap R = P_i$, $i = 1$ and 2, then $Q_1 \subset Q_2$ is a pair of adjacent primes of T.*

Theorem 4.2. *Let T be an integral extension of a domain R. Suppose R is a 1-dimensional strong S-ring. Then T is a strong S-ring.*

Proof. Since T is a integral over R, $\dim T = \dim R = 1$, therefore it is enough to prove T is an S-domain. Let Q be a height 1 prime of T and $P = Q \cap R$, then $P \neq (0)$ and $\operatorname{ht} P \geq 1$. But R is 1-dimensional so $\operatorname{ht} P$ must be exactly 1. Now R is an S-domain hence $PR[X]$ is a height 1 prime of $R[X]$. Also T integral over R implies $T[X]$ is integral over $R[X]$. Moreover, $QT[X]$ lies over $PR[X]$ so that $\operatorname{ht} QT[X] \leq 1$. But $1 \leq \operatorname{ht} QT[X] \leq 2$ always holds so that $\operatorname{ht} QT[X] = 1$; consequently, T is an S-domain.

Corollary 4.3. *Let R be a strong S-domain, P a prime ideal of R of depth ≤ 1. If T is an integral extension of R and Q is a prime ideal of T such that $Q \cap R = P$, then T/Q is a strong S-domain.*

Proof. R/P is a 1-dimemsional strong S-domain and T/Q is an integral extension of R/P, therefore by Theorem 4.2 T/Q is a strong S-ring.

Corollary 4.4. *Let R be a strong S-domain of dimension 2 and P be a prime ideal of height 1. If T is an integral extension of R and Q is a prime ideal of T such that $Q \cap R = P$, then T/Q is a strong S-ring.*

Proof. Since R/P is a domain of dimension ≤ 1 and T/Q is integral over R/P, Corollary 4.3 gives T/Q is a strong S-ring.

Theorem 4.5. *Let R be a 2-dimensional integral domain and T an integral extension of R such that T is an S-domain. Then if R is a strong S-ring, T is a strong S-ring.*

Proof. Let $Q_1 \subset Q_2$ be a pair of adjacent primes of T and $Q_i \cap R = P_i$, $i = 1, 2$. Without loss of generality we may assume $Q_1 \neq (0)$, since if $Q_1 = (0)$, then Q_2 is a height 1 prime of T and by hypothesis T is an S-domain, therefore $(0) \subset Q_2[X]$ are adjacent primes in $T[X]$. Now $\operatorname{ht} P_i \geq \operatorname{ht} Q_i$ and $0 \neq Q_1 \subset Q_2$ are adjacent primes, hence ht $Q_i = i$ for $i = 1, 2$. But then ht $P_2 \leq 2$ since dim $R = 2$, it follows ht $P_2 = 2$. As $P_1 \subset P_2$, ht P_1 has to be 1. Thus $P_1 \subset P_2$ are adjacent primes of R. But R is a strong S-ring implies $P_1[X] \subset P_2[X]$ are adjacent primes of $R[X]$. Since $Q_1[X] \cap R[X] = P_i[X]$, $i = 1, 2$, by Lemma 4.1 $Q_1[X] \subset Q_2[X]$ are adjacent primes of $T[X]$. Thus T is a strong S-ring.

Theorem 4.6. *Let R be a domain and T an integral extension of R. Then if T is a strong S-domain, R is a strong S-domain.*

Proof. By passage to homorphic images, it is enough to prove the following:

If T is integral over R and T is an S-domain, then R is an S-domain.

So, let P be a prime ideal of R of height 1. Then by Theorem 38 [10], $1 \leq \text{ht } P[X] \leq 2$. If ht $P[X] < 2$, then ht $P[X] = 1$ and we are through. If ht $P[X] = 2$ then, as $T[X]$ is integral over $R[X]$ is integral over $R[X]$, there exists a prime ideal Q^* of $T[X]$ such that ht $Q^* = 2$ and $Q^* \cap R[X] = P[X]$. Let $Q = Q^* \cap T$ in T then $Q \neq (0)$ and ht $Q \leq 1$, $Q^* \cap R = Q^* \cap R[X] \cap R = P[X] \cap R = P$, $Q \cap R = P$, and $P \neq (0)$. It now follows that ht $Q = 1$. As T is an S-domain, ht $Q[X]$ is 1. Hence $Q[X] \subset Q^*$. But then $P[X] \subseteq Q[X] \cap R[X] \subseteq Q^* \cap R[X]$. Therefore $P[X] \subseteq Q[X] \cap R[X] \subseteq P[X]$ implies $Q[X] \cap R[X] = P[X]$. Thus $Q[X]$ and Q^* both lie over $P[X]$ which is not possible by INC. Thus R is an S-domain.

Corollary 4.7. *If the integral closure $\bar{R}$ of a domain R in its quotient field K is a strong S-domain, then R is a strong S-domain.*

Corollary 4.8. *Let R be a 1-dimensional strong S-domain and suppose that $X = \{X_1, \ldots, X_n\}$ is a finite set of indeterminates over R. Then R[X] is a strong S-domain.*

Proof. Since R is a 1-dimensional strong S-domain, dim $R[X_1] = 2$ so that the integral closure of R is a Prüfer domain by Theorem 30.14 of [7]. Thus, $R[X_1, \ldots, X_n]$ is a strong S-domain by Theorem 3.5 and Theorem 4.6.

Theorem 4.9. *Let R be an S-domain and T an integral extgension of R. Then, if (R, T) satisfy the* GD-*property* [5], *T is also an S-domain.*

Proof. Suppose Q is a height 1 prime of T and $Q \cap R = P$ then ht $P = 1$ by GD and LO. Now R is an S-domain, thus ht $P[X]$ in $R[X]$ is also 1. As $T[X]$ is integral over $R[X]$ and $Q[X] \cap R[X] = P[X]$, ht $Q[X] \leq 1$. As $Q[X] \neq 0$, ht $Q[X] = 1$. Thus T is an S-domain.

Corollary 4.10. *If R is an integrally closed S-domain and T is integral extension of R, then T is an S-domain.*

Corollary 4.11. *If R is an S-domain and T is a flat R-module such that T is integral over R, then T is also an S-domain.*

Proof. The GD-property holds.

Proposition 4.12. *If R is a* GD-*strong S-domain and T is integral over R then T is a strong S-ring.*

Proof. Let Q be any prime ideal of T and $Q \cap R = P$, then T/Q is integral over R/P and R/P is GD [5]. Therefore T/Q is an S-domain by Theorem 4.6. Since Q was any prime ideal of T it follows T is a strong S-ring.

Theorem 4.13. *Let R be a strong S-domain, T an integral extension of R such that (R, T) satisfy the* GB-*property* [15]. *Then T is a strong S-ring.*

Proof. Let $P^* \subset Q^*$ be a pair of adjacent primes of T and $P^* \cap R = P$, $Q^* \cap R = Q$. Then by definition of the GB-property $P \subset Q$ is a pair of adjacent primes of R. But R is a strong S-ring so that $P[X] \subset Q[X]$ is a pair of adjacent primes of $R[X]$. hence by Lemma 4.1 $P^*[X] \subset Q^*[X]$ are adjacent primes in $T[X]$. Thus T is a strong S-ring.

Corollary 4.14. *Suppose that R is a strong S-domain. Moreover, suppose R is a* GB-*ring. Then $\bar{R}$ is a strong S-ring, where $\bar{R}$ denotes the integral closure of R.*

Theorem 4.15. *Let R be a domain with quotient field K and $\bar{R}$ its integral closure in K. If $\bar{R}$ is a PVMD and T any integral extension of R then T is an S-domain.*

Proof. Let Q be a height 1 prime of T. To prove $Q[X]$ is a height 1 prime of $T[X]$, we use the following result of Seidenberg [17]: $QT[X]$ is a height 1 prime if each prime $\bar{Q}$ of $\bar{T}$ the integral closure of T in the quotient field L of T such that $\bar{Q} \cap T = Q$, is such that $\bar{T}_{\bar{Q}}$ is a valuation ring. So let $\bar{Q}$ be a prime ideal of $\bar{T}$ such that $\bar{Q} \cap T = Q$. Since ht $Q = 1$, ht $\bar{Q} = 1$ also. Furthermore, $\mathrm{ht}(\bar{Q} \cap \bar{R}) = 1$, since the GD-property holds for $(\bar{R}, \bar{T})$. Thus $\bar{Q} \cap \bar{R}$ is a t-ideal of $\bar{R}$ and $\bar{R}$ is a PVMD. It follows that $\bar{R}_{\bar{R} \setminus (\bar{Q} \cap \bar{R})}$ is a valuation ring of rank 1. If $S = \bar{T}_{\bar{R} \setminus (\bar{Q} \cap \bar{R})}$, then S in integral closure of $\bar{R}_{\bar{R} \setminus (\bar{Q} \cap \bar{R})}$ in L and $\bar{Q}S$ is a maximal ideal of S. Moreover, S is a Prüfer domain being the integral closure of a valuation ring and therefore $S_{\bar{Q}S}$ is a valuation ring of rank 1. But

$$S_{\bar{Q}S} = [\bar{T}_{\bar{R} \setminus (\bar{Q} \cap \bar{R})}]_{\bar{Q}\bar{T}_{\bar{R} \setminus (\bar{Q} \cap \bar{R})}} = \bar{T}_{\bar{Q}}$$

so that $\bar{T}_{\bar{Q}}$ is a valuation ring. Hence $Q[X]$ is a height 1 prime of $T[X]$ and T is an S-domain.

Remark 4.16. The proof of Theorem 4.15 only required R to be a P-domain in the terminology of [13].

We have seen in Theorem 4.15 that if the integral closure of a domain R is a PMVD, then any integral extension of R is an S-domain, in particular R itself is an S-domain. In fact, if R were a PVMD to begin with and P a height 1 prime, then R_P is a valuation ring of rank 1 and $PR_P[X]$ is also a height 1 prime of $R_P[X]$ as R_P is a strong S-domain so an S-domain. Hence $PR_P[X] \cap R[X] = PR[X]$ is also a height 1 prime in $R[X]$. Thus a PVMD is an S-domain. We show by an example that a PVMD may not be a strong S-ring. The construction of this example is due to G. Evans.

Example 4.17. Let R be a domain which is not a strong S-ring. Consider $T = Z/(p)$ or $T = Z$ according as the characteristic of R is a nonzero prime integer p or zero. If $\{X_d\}_{d\in R}$ is the set of indeterminates over R, indexed by elements of R then consider $T[\{X_d\}_{d\in R}] = S$. There is a homomorphism $f: S \xrightarrow{\text{onto}} R$ so $S/\ker f = R$. Now S is a PVMD (in fact a Krull domain), but its homomorphic image $S/\ker r$ is not a strong S-ring. Thus, S cannot be a strong S-ring.

Proposition 4.18. *Suppose that T is an integral domain integral over a Prüfer domain R. Then T and $T[X_1, \ldots, X_n]$ are strong S-domains.*

Proof. Let $\bar{R}$ be the integral closure of R in the quotient field of T. Then $\bar{R}$ is a Prüfer domain integral over T. The conclusion follows immediately from Theorems 3.5 and 4.6.

The following corollary is also immediate.

Corollary 4.19. *Suppose that R is a domain with quotient field K and that $\bar{R}$, the integral closure of R in K, is a Prüfer domain. Then R and each domain integral over R is a strong S-domain.*

Proposition 4.20. *Suppose that R is a Noetherian domain, and that T is a domain integral over R. Then T and $T[X_1, \ldots, X_n]$ are strong S-domains.*

Proof. We need only show that T is a strong S-domain for $T[X]$ is integral over the Noetherian domain $R[X]$. If Q is any prime ideal of T and $P = Q \cap R$, then T/Q is integral over R/P and R/P is Noetherian. Hence it is enough to prove T is an S-domain. Now R Noetherian implies that the integral closure $\bar{R}$ in the quotient field of R is a Krull domain by the theorem of Mori and Nagata [6]. Then Theorem 4.15 implies that T is an S-domain.

Corollary 4.21. *Suppose that R is a Noetherian domain and that $\bar{R}$ is the integral closure of R in the quotient field of R. Then $\bar{R}$ is a strong S-domain.*

Note that the ring $\bar{R}$ in Corollary 4.21 is in fact a Krull domain [6] but Example 4.17 show that an arbitrary Krull domain may not be a strong S-domain.

Proposition 4.22. *Suppose that R is a coherent domain and that T an integral extension of R. Then, if $\bar{R}$, the integral closure of R, is a finite R module, then T is an S-domain and consequently R is an S-domain.*

Proof. Since R is coherent and $\bar{R}=R[a_1,\ldots,a_n]$ is a finite R-module, it follows by Corollary 1.4 [10] that R is coherent. Thus $\bar{R}$ is an integrally closed coherent domain and hence a PVMD. It follows by Theorem 4.16 that T is an S-domain and by Proposition 4.6 $\bar{R}$ is an S-domain.

The following example shows that a strong S-domain may not be coherent.

Example 4.23. Let V be a nontrivial valuation ring with quotient field L and V is of the form $V=K+M$, K being a subfield of L and M the maximal ideal of V. Let R be a subring of K which is a Prüfer domain and k its quotient field. Also suppose that K/k is an algebraic extension but $[K:k]$, the degree of K over k, is not finite, then $R_1=R+M$ cannot be coherent. (Or else we may suppose that M is a nonfinitely generated ideal of R_1, then also R_1 is not coherent.) We shall prove in the next section that R_1 is a strong S-ring.

5. Strong S-rings and $D+M$ construction [9]

Let V be a nontrivial valuation ring with quotient field L, and assume that V is of the form $K+M$, where K is a field and M is the maximal ideal of V. Let R be a domain that is a proper subring of K, and let $R_1=R+M$. Also suppose that k is the quotient field of R.

Theorem 5.1. *R_1 is a strong S-ring if and only if R is a strong S-ring and K/k is an algebraic extension of k.*

Proof. Let R_1 be a strong S-domain. Since $R\cong R_1/M$, R is a strong S-ring for homomorphic images of strong S-rings are themselves strong S-rings. By Theorem 5.4 in [9], $\dim R_1[X]=\dim R[X]+\dim V+\inf(d,1)$ where d denotes the transcendence degree of K/k. But R_1 and R are both strong S-rings, it follows that

$$\dim R_1[X]=\dim R_1+1 \quad\text{and}\quad \dim R[X]=\dim R+1.$$

Hence $\dim R_1+1=\dim R+1+\dim V+\inf(d,1)$, but $\dim R_1=\dim R=\dim V$ by the Theorem 2.1 [9]. So $\dim R_1+1=\dim R+1+\inf(d,1)$. This implies that $\inf(d,1)=0$ and therefore $d=0$. Therefore K/k is algebraic.

Conversely, suppose R is a strong S-ring and K/k is an algebraic extension. Let $Q_1\subset Q_2$ be adjacent primes of R_1. Then three different cases may arise.

Case 1. If $M\subseteq Q_1,Q_2$ then $Q_i=P_i+M$, where P_i are prime ideals of R. Clearly $P_1\subset P_2$ are adjacent primes of R, for if $P_1\subsetneq P\subsetneq P_2$, then $Q=P+M$ is a prime ideal

of R_1 and $Q_1 \subseteq Q \subseteq Q_2$. But then $Q_1 \subset Q_2$ are adjacent primes of R_1, either $Q_1 = Q$ or $Q_2 = Q$; consequently $P = P_1$ or $P = P_2$. Now R is a strong S-ring so that $P_1[X] \subset P_2[X]$ are adjacent primes of $R_1[X]$ because $Q_i[X] \cap R[X] = P_i[X]$.
Case 2. If $Q_1 \subseteq M$ and $M \subseteq Q_2$, then as $Q_1 \subset Q_2$ is a pair of adjacent primes of R_1 either $Q_1 = M$ or $Q_2 = M$. In either case the argument in case 1 completes the proof.
Case 3. If Q_1, Q_2 are both contained in M, then they are both prime ideals of V, but V is a valuation ring and hence a strong S-ring. Therefore $Q_1[X] \subset Q_2[X]$ are adjacent primes.

Theorem 5.2. *Let R be a domain with quotient field K and $R_1 = R + XK[X]$. Then R_1 is a strong S-ring if and only if R is a strong S-ring.*

Proof. Let R_1 be a strong S-ring. Then, as $R_{1/XK[X]} \simeq R$ it follows that R is a strong S-ring.

Conversely, let R be a strong S-ring and $Q_1 \subset Q_2$ be adjacent primes of R_1. We consider the following two cases:
Case 1. If Q_1 and Q_2 are not both principal, then $Q_i = P_i + XK[X]$, where P_i are prime ideals of R. Clearly $P_1 \subset P_2$ is a pair of adjacent primes of R. But R is a strong S-ring, so $P_1[Y] \subset P_2[Y]$ are adjacent primes in $R[Y]$. As $Q_i[Y] \cap R[Y] = P_i[Y]$, $Q_1[Y] \subset Q_2[Y]$ are adjacent primes in $R_1[Y]$.
Case 2. If Q_i is not a principal ideal but Q_2 is a principal ideal then Q_2 is a height 1 maximal ideal of R_1 such that $Q_2 \cap R = (0)$, hence $Q_1 \cap R = (0)$ forcing $Q_1 = (0)$. If $Q_2 = f(x)R_1$, where $f(x)$ is irreducible in $K[X]$ and $f(0) = 1$, then $Q_2 = f(x)R_1 = Q_2K[X] \cap R_1$ and Q_2 is a height 1 prime ideal of $K[X]$. Now $K[X]$ is a strong S-ring, so that $Q_2[Y]$ must be of height 1 hence $(0) \subset Q_2[Y]$ are adjacent primes.

We now examine the behaviour of the strong S-property in two other constructions, which are similar to the previous '$R+M$' construction. The details of these constructions are given in [8] and [12].

Let L be a field and K a subfield of L and $\{V_i\}_{i=1}^n$ a finite collection of nontrivial valuation rings of L such that (i) $V_i \not\subset V_j$ for $i \neq j$; and (ii) each V_i is of the form $K + M_i$, M_i the maximal ideal of V_i. Let D_i be subrings of K with quotient fields k_i and set $J_i = D_i + M_i$, $J = \bigcap_{i=1}^n J_i$ and $V = \bigcap_{i=1}^n V_i$. If $N_i = V \cap M_i$, then $H_i = M_i \cap J$ and $M = \bigcap_{i=1}^n H_i$ and then

$$M = \bigcap_{i=1}^{n} (J \cap M_i) = J \cap \left(\bigcap_{i=1}^{n} M_i \right) = \bigcap_{i=1}^{n} M_i.$$

Also $M = \bigcap_{i=1}^n N_i$. Denote by C_1 the set of all primes that are contained properly in some H_i and C_2 the set of all primes of J that contain some H_i.

Theorem 5.3. *J is a strong S-ring if and only if each D_i is a strong S-ring, and K/k_i is an algebraic extension of k_i for each i.*

Proof. Suppose first that J is a strong S-ring. To prove that each D_i is a strong S-ring, we first note that by Theorem 4.10 of [1] $J_i/M_i = D_i = J/H_i$. Moreover, J_i is a quotient ring of J. Now since J is a strong S-ring, it follows J_i is a strong S-ring. But then D_i is a homomorphic image of J_i, hence D_i is a strong S-ring and by Theorem 4.1 K/k_i is a algebraic over k_i.

Conversely, let D_i be a strong S-ring and K/k_i be algebraic over k_i, then by Theorem 5.1 J_i is a strong S-ring for each i. Moreover, for each maximal ideal M of J, M contains a unique H_i and there is a prime ideal P_i of J_i such that $P_i \cap J = M$. Moreover, each J_i is a quotient ring of J and $J_M = (J_i)_{P_i}$. Thus, J_M is a strong S-domain for each maximal ideal M of J and by Proposition 2.3, J is a strong S-domain.

Remark 5.4. Observe that in [3] Example 3 shows that if $D[X]$ is a strong S-ring, $(D+M)[X]$ need not be a strong s-ring.

Let V_i be independent valuation domains with quotient field L and K_i be the residue field of V_i for all i, $1 \le i \le n$. Let K be embedded in $\sum_{i=1}^n K_i$ via the diagonal map and D a subring of K. Set $J_i = D + M_i$, $J = \bigcap_{i=1}^n J_i$, $V = \bigcap_{i=1}^n V_i$. Then

$$J = \bigcap_{i=1}^{n} J_i = D + \bigcap_{i=1}^{n} M_i$$

is a domain with quotient field L. Assume k is quotient field of D. Then we have:

Theorem 5.5. *J is a strong S-domain if and only if D is a strong S-domain and K/k is algebraic.*

Proof. Let D be a strong S-ring and K/k an algebraic extension. Suppose $P_1 \subset P_2$ are adjacent primes of J. Now each prime ideal of J compares with $I = \bigcap_{i=1}^n M_i$ [12]. Once again three cases arise.

Case 1. If $I \subseteq P_1 \subset P_2$, then $I = \bigcap M_i \subseteq P_1$ implies there exists an i such that $M_i \subseteq P_1$. Therefore $M_i \subseteq P_1 \subset P_2$ and there exist prime ideals Q_1 and Q_2 of D such that $P_1 = Q_1 + I$, $P_2 = Q_2 + I$. Since $P_1 \subset P_2$ are adjacent primes, $Q_1 \subset Q_2$ are also adjacent primes of D, it then follows for each fixed i that $Q + M_i \subset Q_2 + M_i$ are adjacent primes in J_i. But $Q_j + M_i \subseteq P_j$ as $M_i \subset P_j$, $j = 1, 2$, and $P_j = Q_j + I \subseteq Q_j + M_i$. Therefore $P_j = Q_j + M_i$. Now J_i is a strong S-ring implies $(Q_1 + M_i)[X] \subset (Q_2 + M_i)[X]$ are adjacent primes of $J_i[X]$. As $(Q_j + M_i)[X] = P_j[X]$ therefore $P_i[X] \subset P_2[X]$ are adjacent primes of $J[X]$.

Case 2. If $P \subseteq I \subseteq P_2$, then $I \subseteq P_2$ implies there exists an integer i such that $M_i \subseteq P_2$. Therefore $P_1 \subseteq M_i \subsetneqq P_2$, it now follows that $P_1 \subseteq M_i \cap (\bigcap_{k \neq i} J_k) \subseteq P_2$. But $P_1 \subset P_2$ are adjacent primes of J, therefore either

$$P_1 = M_i \cap \left(\bigcap_{k \neq i} J_k \right) \quad \text{or} \quad P_2 = M_i \cap \left(\bigcap_{k \neq i} J_k \right).$$

If $P_1 = M_i \cap (\bigcap_{k \neq i} J_k)$, then as $I \subseteq M_i \cap (\bigcap_{k \neq i} J_k)$, $I = P_1$. Hence $P_1 = M_i$. If $P_2 = M_i \cap (\bigcap_{k \neq i} J_k)$, then as $M_i \subset P_2 = M_i \cap (\bigcap_{k \neq i} J_k) \subset M_i$, $P_2 = M_i$. Thus either $P_1 = M_i$ or $P_2 = M_i$ and case 1 occurs. Therefore $P_1[X] \subset P_2[X]$ are adjacent primes.

Case 3: If $P_1 \subset P_2 \subseteq I = \bigcap_{i=1}^n M_i$, then P_1 and P_2 are ideals in each J_i and prime ideals of V_i for each i. Because if $y \in V_i$ and $x \in P_j$ choose $m_i \in M_i \setminus P_j$, then $xy \in J_i$ and $m_i \in J_i$ implies $xym_i = x(ym_i) \in P_j$ but $m_i \notin P_j$ implies $xy \in P_j$ thus P_i and P_2 are prime ideals of V_i. But then V_i is a valuation ring and hence a strong S-ring. Thus $P_1[X] \subset P_2[X]$ are adjacent primes of $J[X]$.

Conversely, let J be a strong S-ring, then since D is homomorphic image of J, D is a strong S-ring. Now by invoking Theorem 5.1 for each fixed i, J_i is a strong S-ring and K/k is an algebraic extension of k.

References

[1] J. Arnold and R. Gilmer, The dimension sequences of a commutative ring, Amer. J. Math. 96 (1974) 385-408.

[2] J. Brewer, The ideal transform and overrings of integral domain, Math. Z. 107 (1968) 301-306.

[3] J. Brewer, P.R. Montgomery, E.A. Rutter and W.J. Heinzer, Krull dimensions of polynomial Rings, Lecture Notes in Math. No. 311 (Springer, Berlin-New York, 1972).

[4] D. Costa, J.L. Mott and M. Zafrullah, The $D + XD_S[X]$ construction, J. Algebra 53 (1976) 423-429.

[5] D. Dobbs, Divided rings and going down, Pacific J. Math. 67(2) (1976) 353-363.

[6] R. Fossum, The Divisor Class Group of a Krull Domain (Springer, New York, 1973).

[7] R. Gilmer, Multiplicative Ideal Theory (Marcel Bekker, New York, 1972).

[8] R. Gilmer, Two constructions of Prüfer domains, J. Reine Angew, Math. 239 (1970) 153-162.

[9] R. Gilmer and E. Bastida, Overrings and divisorial ideals of rings of form $D + M$, Michigan Math. J. 20 (1973) 79-95.

[10] M. Harris, Some results on coherent rings, Proc. Amer. Math. Soc. 18 (1967) 749-753.

[11] I. Kaplansky, Commutative Rings (Allyn and Bacon, Boston, MA, 1974).

[12] J.L. Mott and M. Schexnayder, Exact sequence of semivalue groups, J. Reine Angew. Math. 283/284 (1976) 388-401.

[13] J.L. Mott and M. Zafrullah, On Prüfer v-multiplication domains, Manuscripta Math. 35 (1981) 1-26.

[14] M. Nagata, Finitely generated rings over a valuation ring, J. Math. Kyoto Univ 5(2) (1966) 163-169.

[15] L.J. Ratliff, Going between rings and contractions of saturated chains of primes, Rocky Mountain J. Math. 7(4) (1977) 777-787.

[16] F. Richman, Generalized quotient rings, Proc. Amer. Math. Soc. 16 (1965) 794-799.

[17] A. Seidenberg, On dimension theory of rings, Pacific J. Math. 3 (1953) 513-522.